Modelling the Dynamical Motion of Space Debris

MATHEMATICAL MODELLING ASSIGNMENT 2

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ABSTRACT

Space debris is an ever-increasing issue which poses a huge threat towards space exploration in the future. With space venturing becoming more routine, if something is not done this problem will only intensify. First of all, we begin to explore the possible damage caused as a result of the debris; moving onto a few methods that reduce the quantity of debris and protect spacecrafts in orbit from these detrimental effects. The main research in this essay is aimed towards modelling the motion of a spacecraft in relation to debris in an orbit above the Earth with the ultimate goal to optimise the distance between the two orbital objects and clean the debris by dragging it further towards the atmosphere. As a matter of fact, we can simulate this given a few initial conditions, allowing us to numerically solve the system using the Runge-Kutta 4th order method. Finally, after minimising the distance, we expand on the interactions at a molecular level, occurring between the debris and the atmosphere as it falls back down to Earth.

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1 Introduction

Within a geosynchronous orbit around the Earth, it is estimated that hundreds of millions of debris pieces already exist. Naturally we can categorise debris according to size, however from Earth it's nearly impossible to track debris of only a few inches wide. Man-made debris can arise from almost anything on a spacecraft, this can include launch canisters or even flakes of dust and paint from the body; whilst natural debris occurs in our orbit due to meteors and asteroids [1]. Space debris has caused much concern recently due to the amount of damage it can inflict, which we shall now explore in more detail.

1.1 Direct and Indirect Threats of Debris

The main issue with space debris is the damage caused to active spacecrafts, particularly to the manned crafts as this poses a direct threat to human life. Even relatively small debris, which cannot be remotely tracked, can pierce the shell of a spacecraft; often destroying the craft, which in turn becomes debris itself. If something is not done to prevent the creation of debris, its quantity will increase exponentially, this is known as Kessler Syndrome [2].

On the other hand, many pieces of debris are within a geosynchronous orbit (this type of orbit is defined to have the same orbital period as the celestial object) and may fall back into the Earth's atmosphere, despite much of the debris burning up as it enters the atmosphere. This might present a huge impact should the quantity of debris continue to increase and if one of the many fragments should strike land.

1.2 Reduction Techniques and Palliative Measures

With the vast quantity of debris floating in space the likelihood of a spacecraft colliding with debris is relatively probable, fully quantified in Section (2.6). Which is why efforts are made to collect such debris, as well as mitigate impact effects to a craft. However with private spaceflight becoming increasingly popular and affordable, this problem is only likely exacerbated. Nevertheless we can employ several techniques in an attempt to combat this issue.

1.2.1 Reduction Techniques

The first commonly used method is known as a laser broom [3]. It is essentially a ground-based laser with enough power to pierce the atmosphere and ablate the orbital debris, hence dragging it out of orbit. However to use this method, the debris must be approaching the point in its orbit directly above the ground-laser in order for the laser to take effect. Another use of this is to control the position of debris by altering its speed gradually to avoid collisions. Although in order to transfer sufficient energy the laser must be in contact for several hours [4].

More recently the UK are working on a net that can trap pieces of debris and once again drag them towards the atmosphere. Attached to a spacecraft with thrusters, the net is ejected from the craft using a harpoon.

The final technique explored here is the most important to our studies later on in the report. Spacecrafts equipped with claws to catch debris can be utilised to collect fragments and drag them towards the atmosphere. One problem with this method is that the craft must be in close proximity to the debris in order to remove it [5].

1.2.2 Palliative Measures

With the shell and body of a spacecraft only a few inches thick, it is no wonder that such small shards of debris pose such a large threat. As debris travels at a hypervelocity, faster than the speed of sound, the impact creates a huge amount of stress in that region, which can lead to crack propagation within the material [6].

The best form of protection on a spacecraft is a shield. Specifically made shields, called Whipple bumpers, are attached to a craft with the aim to break apart incoming debris. Therefore the momentum is spread over a larger area and the chassis is more likely to withstand the pressure. However these shields often significantly increase the weight of the craft and hence the fuel needed, so are only used when entirely necessary.

Having seen a few ways in which we can reduce this threat, we shall expand on the physical relation between two orbital objects: a spacecraft and debris. We want to formulate a model that we can simulate in order to describe more about the system.

2 Formulating Our Model

To describe the system we plan to work with, we can use classical mechanics to introduce formulae so that we can begin to develop the system mathematically. Our model will simply contain three objects: the Earth, debris and the spacecraft; where we can model each object's position in relation to one another. Ultimately we want to create a model with adjustable parameters so that we can minimise the distance between two orbital objects about a celestial object.

2.1 Newtonian Mechanics

In order to describe the movement of objects under gravity, we must first consider Newton's laws of motion and gravity. These laws state that the acceleration of an object under motion is directly proportional to the force acting on it, and further that every element with mass m_1 is also attracted to every other element with mass m_2 by a proportional force [7]. Newton summarised these laws by two equations,

$$F = ma, \quad F = -\frac{Gm_1m_2}{r^2}.$$

In terms of our system we can model the force between two objects, most commonly between the Earth and debris, we can suitably ignore the force acting between the two orbiting objects as their masses are relatively negligible. With the idea that in a two-dimensional space we have more than one component, the force in each direction can be modelled as follows,

$$\begin{aligned} m\ddot{x} &= -\frac{GM_Em}{x^2+y^2}\frac{x}{\sqrt{x^2+y^2}} + F_x(t), \\ m\ddot{y} &= -\frac{GM_Em}{x^2+y^2}\frac{y}{\sqrt{x^2+y^2}} + F_y(t). \end{aligned} \tag{2.1}$$

$G = 6.673 \times 10^{-11} Nm^2kg^{-2}$ is Newton's gravitational constant, $M_E = 5.97 \times 10^{24}$ is the mass of the earth and F_x, F_y represent the adjustable forces applied in the x and y directions respectively.

Q1

As a matter of fact, equations (2.1) can be simplified and hence solved if we use a change of variables into polar coordinates. Substituting $x = r\sin(\theta)$, $y = r\cos(\theta)$ we get,

$$\begin{aligned} \ddot{x} &= -\frac{GM_E}{r^2}\sin\theta + \frac{F_x}{m}, \\ \ddot{y} &= -\frac{GM_E}{r^2}\cos\theta + \frac{F_y}{m}. \end{aligned} \tag{2.2}$$

Now using the chain rule twice on both variables x and y we eventually find that,

$$\begin{aligned} \dot{x} &= \dot{r}\sin\theta + r\dot{\theta}\cos\theta, \\ \dot{y} &= \dot{r}\cos\theta - r\dot{\theta}\sin\theta, \end{aligned} \tag{2.3}$$

$$\begin{aligned} \ddot{x} &= (\ddot{r} - r\dot{\theta}^2)\sin\theta + (2\dot{\theta}\dot{r} + r\ddot{\theta})\cos\theta, \\ \ddot{y} &= (\ddot{r} - r\dot{\theta}^2)\cos\theta - (2\dot{\theta}\dot{r} + r\ddot{\theta})\sin\theta. \end{aligned} \tag{2.4}$$

To find these equations explicitly in terms of polar coordinates, we can rearrange for $\ddot{r}$ and $\ddot{\theta}$; in actual fact we can use a linear combination of equations (2.4) which will aid us in eliminating particular terms of (2.4). If we first aim to solve for $\ddot{r}$ we can use the following combination,

$$\begin{aligned}
\ddot{x}\sin\theta+\ddot{y}\cos\theta &= \Big[(\ddot{r}-r\dot{\theta}^2)\sin^2\theta+(2\dot{\theta}\dot{r}+r\ddot{\theta})\cos\theta\sin\theta\Big]+\Big[(\ddot{r}-r\dot{\theta}^2)\cos^2\theta-(2\dot{\theta}\dot{r}+r\ddot{\theta})\sin\theta\cos\theta\Big],\\
&= \ddot{r}-r\dot{\theta}^2,\\
\ddot{r} &= \ddot{x}\sin\theta+\ddot{y}\cos\theta+r\dot{\theta}^2,\\
\ddot{r} &= -\frac{GM_E}{r^2}\sin^2\theta+\frac{F_x}{m}\sin\theta-\frac{GM_E}{r^2}\cos^2\theta+\frac{F_y}{m}\cos\theta+r\dot{\theta}^2,
\end{aligned}$$

$$\boxed{\ddot{r}=-\frac{GM_E}{r^2}+\frac{F_r}{m}+r\dot{\theta}^2.} \tag{2.5}$$

Here, we have also defined $F_r = F_x\sin(\theta)+F_y\cos(\theta)$, writing F_r in terms of our old parameters x and y. Furthermore we can solve $\ddot{\theta}$ from equations (2.4) using the same method. Taking a linear combination to this time eliminate each $(\ddot{r}-r\dot{\theta}^2)$ term, we obtain,

$$\begin{aligned}
\ddot{x}\cos\theta-\ddot{y}\sin\theta &= \Big[(\ddot{r}-r\dot{\theta}^2)\sin\theta\cos\theta+(2\dot{\theta}\dot{r}+r\ddot{\theta})\cos^2\theta\Big]-\Big[(\ddot{r}-r\dot{\theta}^2)\cos\theta\sin\theta-(2\dot{\theta}\dot{r}+r\ddot{\theta})\sin^2\theta\Big],\\
&= 2\dot{\theta}\dot{r}+r\ddot{\theta}. \qquad (2.6)
\end{aligned}$$

However, by (2.2), $$\begin{aligned}
\ddot{x}\cos\theta-\ddot{y}\sin\theta &= -\frac{GM_E}{r^2}\sin\theta\cos\theta+\frac{F_x}{m}\cos\theta+\frac{GM_E}{r^2}\sin\theta\cos\theta-\frac{F_y}{m}\sin\theta,\\
&= \frac{F_\theta}{m}. \qquad (2.7)
\end{aligned}$$

Now we can similarly define F_θ in terms of our new variables, that is $F_\theta = F_x\cos(\theta)-F_y\sin(\theta)$, we shall fully define F_r and F_θ in Section (2.3). Hence by equating and rearranging (2.6) and (2.7) the solution follows,

$$\boxed{\ddot{\theta}=-2\frac{\dot{\theta}\dot{r}}{r}+\frac{F_\theta}{mr}.} \tag{2.8}$$

2.2 Trivial Example

In order to learn more about our system we can envisage the trivial solution, implying that all forces in our system are fixed at zero, that is $F_r = F_\theta = 0$. This suggests that by Newton's second law, the velocity and angular frequency of any orbiting object remain constant. Taking this literally, if an object has no force exerting on it, we can conclude a few statements.

Firstly at an altitude h above the Earth of radius R_E, that $r = r_0 := R_E+h$, since the object does not move further away from the Earth because F_r is fixed at zero. Similarly that $\theta = \omega_0 t+\theta_0$ while the object may rotate about the earth, but with only constant frequency according to Newton's second law and $F_\theta = 0$. ω_0 is defined below.

Under the assumption that the circular movement of an object about the Earth can be modelled using simple harmonic motion, we can state the angular frequency ω_0 of the object,

$$r = r_0\sin(\omega_0 t)\ ,\quad \ddot{r} = -r_0\omega_0^2\sin(\omega_0 t), \tag{2.9}$$

$$\text{by (2.5)}:\ -r_0\omega_0^2\sin(\omega_0 t) = -\frac{GM_E}{r_0^2}+\frac{F_{r_0}}{m}+r_0\dot{\theta}^2. \tag{2.10}$$

As we know $F_r = F_\theta = 0$, we can eliminate the second term in (2.10), also since there is no force in the θ-direction we know $\dot{\theta} = 0$. If we evaluate t at some time $T_0 = \frac{2\pi}{\omega_0}$, which by the theory of harmonic motion is defined as the time for one orbital circuit, (2.10) reduces to give us an explicit formula for the angular frequency,

$$r_0\omega_0^2 = \frac{GM_E}{r_0^2}, \tag{2.11}$$

$$\omega_0 = \sqrt{\frac{GM_E}{r_0^3}}, \text{ at } T_0 = \frac{2\pi}{\omega_0}. \tag{2.12}$$

2.3 Calculating Other Important Quantities

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With more relevance to our own model, assuming that a piece of debris circulates the earth, we can calculate the average speed (which is also it's instantaneous speed, since the forces are still zero). As the circumference or distance travelled in an orbit is $2\pi r_0$ and that T_0 is the orbital period, we can say that it travels with speed v, that is,

$$\boxed{v = \frac{2\pi r_0}{T_0} = \sqrt{\frac{GM_E}{r_0}}.} \tag{2.13}$$

Up to now we have naïvely defined the forces F_r and F_θ. However these forces in fact arise from the thrusters on the spacecraft, where in reference to our model F_r is the radial force and F_θ is the force of the thrusters acting parallel to the Earth's surface. These variables are used to slightly alter the current position of the craft, independent of its orbit. Two important quantities which we shall use later on in Section (3) is the time at which the thrusters stop, as we only have limited fuel, we can denote this by t_{thrust}. Furthermore the fuel consumption of the craft can be defined by,

$$\text{Fuel} = \big(|F_r| + |F_\theta|\big) \cdot t_{thrust}. \tag{2.14}$$

Example : At a height $h = 330km$ above the surface, we can calculate the orbital period for debris circling the Earth. $R_E = 6.37 \times 10^6 m$, $G = 6.673 \times 10^{-11} Nm^2kg^{-2}$, $M_E = 5.97 \times 10^{24} kg$.

$$\begin{aligned} r_0 &= 330000 + 6370000\ ,\ \ r_0^3 = 3.01 \times 10^{20}\ \ (3\ s.f.), \\ \omega_0 &= \sqrt{\frac{6.673 \times 10^{-11} \cdot 5.97 \times 10^{24}}{3.01 \times 10^{20}}} = 1.15 \times 10^{-3}, \\ T_0 &= \frac{2\pi}{1.15 \times 10^{-3}} = 5460 \text{ seconds} \approx 91 \text{ minutes}\ \ (3\ s.f.). \end{aligned}$$

Now at a height of $435km$ above the surface,

$$\begin{aligned} r_0 &= 435000 + 6370000\ ,\ \ r_0^3 = 3.15 \times 10^{20}\ \ (3\ s.f.), \\ \omega_0 &= \sqrt{\frac{6.673 \times 10^{-11} \cdot 5.97 \times 10^{24}}{3.15 \times 10^{20}}} = 1.12 \times 10^{-3}, \\ T_0 &= \frac{2\pi}{1.12 \times 10^{-3}} = 5590 \text{ seconds} \approx 93 \text{ minutes}\ \ (3\ s.f.). \end{aligned}$$

Both of our answers seem a reasonable value for the orbital period at these heights, in fact we can confirm our answer as the ISS has a time period of 92.66 minutes [8].

2.4 Introduction of a Spacecraft

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Now we want to consider the introduction of a spacecraft in our model, with a non-trivial solution. We can in fact use the polar coordinate system once again to describe the movement of both debris and a spacecraft, and hence can define the parameters r and θ as follows,

$$r_d = r_0,\ \theta_d = \omega_0 t \ \text{ and }\ r_s = r_0 + z(t),\ \theta_s = \omega_0 t + \phi(t). \tag{2.15}$$

The functions of times z and ϕ arise in the spacecraft coordinates as a result of the thrusters. One crucial quantity in our system will be the distance between the debris and the spacecraft. This is particularly important as we want to minimise the distance between them, so the debris can be removed. Using the definitions of the parameters above, we can derive an equation for this, which we shall denote L.

By the cosine rule, we have that,

$$\begin{aligned} L^2 &= r_d^2 + r_s^2 - 2r_d r_s \cos(\theta_d - \theta_s), \\ &= r_0^2 + (r_0 + z(t))^2 - 2r_0(r_0 + z(t))\cos(\theta_d - \theta_s), \\ &= 2r_0^2 + 2r_0 z(t) + z(t)^2 - \left(2r_0^2 + 2r_0 z(t)\right)\cos\phi(t), \end{aligned} \tag{2.16}$$

$$\boxed{L = \sqrt{(1-\cos\phi)(2r_0^2 + 2r_0 z) + z^2}.} \tag{2.17}$$

In order to prove this describes the distance between the two objects, we can show some simple cases below where the answers are trivial.

① $\boxed{z = 0,\ \phi = 0}$: $L^2 = 2r_0^2 - 2r_0^2 = 0$. Since both the difference in θ and z is 0, we have that both points are the same, hence the distance between them is 0.	② $\boxed{z = a,\ \phi = 0}$: $L^2 = 2r_0^2 + 2r_0 a + a^2 - (2r_0^2 + 2r_0 a) = a^2$. This time the objects have the same angle, implying that the distance will only differ by the difference in lengths of the vectors to each point. That is, $(r_0 + a) - r_0 = a$, $L = a$.
③ $\boxed{z = 0,\ \phi = a}$: $L^2 = 2r_0^2 - (2r_0^2 \cos a) = 2r_0^2(1 - \cos a)$. By using the cosine rule for an angle of a we obtain the same answer as our formula, $L = \sqrt{2r_0^2(1-\cos a)}$.	④ $\boxed{z = 0,\ \phi = \frac{\pi}{2}}$: $L^2 = 2r_0^2$. As the angle between the two objects is $\frac{\pi}{2}$ we can apply Pythagoras's Theorem and since $z = 0$, we have $L = r_0\sqrt{2}$.
⑤ $\boxed{z = a,\ \phi = \pi}$: $L^2 = 2r_0^2 + 2r_0 a + a^2 + (2r_0^2 + 2r_0 a) = (2r_0 + a)^2$. While the set of both points, including the origin is collinear, we have that the points lie in a straight line. Therefore $L = 2r_0 + a$, which is also the solution of our formula.	⑥ $\boxed{z = a,\ \phi = \frac{\pi}{2}}$: $L = \sqrt{2r_0^2 + 2r_0 a + a^2}$ Using trigonometry, we logically find the same answer.

Table 1: Trivial Examples

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2.5 Impact of Debris

As explained earlier in Section (1.1) the impact of debris poses a great threat to the Earth. As a matter of fact, we can calculate the impact velocity of debris under a few general assumptions. We shall first assume that a $5cm^3$ cube of aluminium of mass $m = 0.34kg$ exists in a geosynchronous orbit, although in actual fact the impact speed is entirely independent of its mass and size, excluding the influence of air resistance. Now we can derive the orbit for the Earth.

By equating Newton's forces and the centrifugal force, we have,

$$\frac{GM_E m}{r^2} = \frac{mv^2}{r}.$$

For a circular orbit we now have that,

$$F = \frac{m(\frac{2\pi r}{t})^2}{r} = \frac{4\pi^2 rm}{t^2},$$

$$\frac{GM_E}{r^2} = \frac{4\pi^2 r}{t^2},$$

$$r = \sqrt[3]{\frac{GM_E t^2}{4\pi^2}}.$$

Using a stellar day, that is $t = 86164$ seconds, from our formula $r = 42.156 \times 10^6$m, which is equivalent to $35,786$km above the Earth's surface. The velocity v_{geo} at this height is equal to, $\frac{2\pi r}{t} = \frac{2\pi \cdot 4.2156 \times 10^7}{86164} = 3074 m/s$. By suggesting that all potential energy is transferred to kinetic energy during its flight, we have an impact speed $v = 10000 m/s$. However this is only an upper bound as energy is also converted to heat and sound, and the debris' terminal velocity may have been achieved. In fact we might expect the impact speed to be almost half this! We will expand on this theory later in Section (5), whilst introducing the density of the atmosphere and interaction between molecules.

2.6 Debris Density

At an altitude of $400km$ the density of space debris is estimated to be roughly $10^{-18} m^{-3}$, implying that the probability of a spacecraft of surface area $2500m^2$ colliding with debris at an orbital speed of $7671 m/s$ (fully derived later in equation (4.1)) is given by:

$$p = \rho_d \, A \, v,$$
$$p = 10^{-18} \cdot 2500 \cdot 7671 = 1.918 \times 10^{-11} s^{-1}.$$

Furthermore, that the number of collisions theoretically occurring on average is the probability times the number of seconds in a year,

$$N = 1.918 \times 10^{-11} \cdot 3.153610^7 = 0.0006.$$

This figure is much less than one might expect, especially over an entire year. Yet with the popularity of space exploration increasing, the number of spacecraft in a low-earth orbit will rise, along with the amount of debris. In turn the probability will rise according to the Kessler Syndrome problem, first introduced in Section (1.1).

3 Evaluating the Model

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Once we have set up our model we can begin to solve it. We will use the Runge-Kutta 4^{th} order method to iteratively solve the ordinary differential equations found in Section (2.1). Unlike in Section (2), we can now suggest that the spacecraft has external force components F_r and F_θ, this then implies that r is not fixed and can vary; likewise, the rate of change of θ is not necessarily constant. We can represent these additional forces by $r = r_0 + z(t)$ and $\theta = \omega_0 t + \phi(t)$, as previously shown in (2.15).

Moreover, we can now begin to solve our differential equations. By substituting the new polar equations for the spacecraft into (2.5) and (2.8) we have,

$$\ddot{z} = -\frac{GM_E}{(r_0+z)^2} + (r_0+z)(\omega_0+\dot{\phi})^2 + \frac{F_r}{m}, \tag{3.1}$$

$$\ddot{\phi} = -2\frac{(\omega_0+\dot{\phi})\dot{z}}{(r_0+z)} + \frac{F_\theta}{m(r_0+z)}. \tag{3.2}$$

Next, reducing the system of second order differential equations into a set of four first order equations can make the problem a lot simpler, if we let $\dot{z} = v_z$ and $\dot{\phi} = v_\phi$ we have that,

$$\dot{v}_z = -\frac{GM_E}{(r_0+z)^2} + (r_0+z)(\omega_0+v_\phi)^2 + \frac{F_r}{m}, \tag{3.3}$$

$$\dot{v}_\phi = -2\frac{(\omega_0+v_\phi)v_z}{(r_0+z)} + \frac{F_\theta}{m(r_0+z)}. \tag{3.4}$$

3.1 Runge-Kutta 4th Order Method

This is a method that gives numerical approximations to the solutions of four ordinary differential equations, it is more complex than both Euler's method and the second order Runge-Kutta method. In our case we can use the fourth order method as it gives a better approximation of the true value. It works by integrating four different slopes and estimates at the midpoint of each slope [9], we can iterate this using *Python* over many time intervals.

Implementing this into our model, we can iterate over a vast range of time intervals, calculating the minimum distance at each integration step using (2.16) given the z and ϕ values from Runge-Kutta. Saving each distance after every iteration, we can obtain a global minimum over a large time interval.

4 Manoeuvring Towards the Debris

Q5

Although manoeuvring in a straight line towards the target might seem the easiest course to reach the debris, it is not always possible. This is because the relative altitude is also increasing at any stage in its orbit. Aiming directly at the target will cause the craft to miss the target and reach a higher altitude than required. It is also important that the debris keeps its speed, as if the centrifugal force outweighs the gravitational force the debris will begin to fall to Earth. In fact any small change in speed will alter its orbit, which is why many orbits are instead elliptical. At an altitude of $400km$,

$$\frac{GM_E m}{r^2} \leq \frac{mv^2}{r},$$

$$v \geq \sqrt{\frac{6.673 \times 10^{-11} \cdot 5.97 \times 10^{24}}{6770 \times 10^{3}}} = 7671 m/s. \tag{4.1}$$

4.1 Three Initial Cases

To restate the initial conditions of our system, the debris travels on a reference orbit at and altitude of $400km$, with the craft initially $1km$ below and roughly $2km$ behind its orbit. The debris and spacecraft both have coordinates as described by equations (2.15).

If we now assume that the spacecraft has a maximum of $1.5 \times 10^4\ kg\ m/s$ of fuel, defined previously in Section (2.3), we can split this into several cases with the aim to optimise the distance between the craft and debris. We can now consider the following cases: $F_r = 50$, $F_\theta = 100$, $t_{thrust} = 100$; $F_r = 25$, $F_\theta = 50$, $t_{thrust} = 200$ and $F_r = 10$, $F_\theta = 20$, $t_{thrust} = 500$. In fact evaluating these three cases using our model, we find that the last case reaches closest to the debris (279.7m) within one orbital time period, with all cases obtaining their minimum distance between 8 and 12 minutes. However this minimum distance is indeed far too large for the craft to collect the debris, instead we can assume that the debris can be successfully collected if the distance is reduced to $1m$, but our aim remains to minimise this distance.

4.2 Optimising the Distance

Taking the first case as above we can vary the value of t_{thrust} between 0 and 500 and observe the minimum distance between the two objects within 15 minutes after thrusting. Using our model, we notice that as t_{thrust} increases, the minimum distance in that particular orbit decreases. Excluding values where no minimum exists we observe that as t_{thrust} reaches 350.0 the distance tends to 220.627, which is obtained at $t = 350.0$. In fact for all values of t_{thrust} greater than 350.0 we have the same minimum point.

4.2.1 Existence of Minima

Despite this, for several values of t_{thrust} we find that a minimum does not exist and this can happen for several reasons. First of all, the speed of the object can become sufficiently large so that it reaches its escape velocity, yet with a calculation we can estimate the velocity at this height to be roughly $11kms^{-1}$, which by further calculations from our model is very unlikely to be achieved with the forces involved. In fact assuming the burners can last for 500 seconds, we would need a force over 80 kN, but this is almost impossible to produce from a single burner.

Moreover, two more likelier reasons that a minimum might not exist is because the minimum for a particular orbit occurs before a time $t = t_{thrust}$, or whilst the craft is thrusting. The last minimum problem we encounter here is when a minimum occurs after $t = (900 + t_{thrust})$, meaning that no minimum has yet been obtained, hence we can instead bound the decreasing values of the distance below by an infimum, which is the shortest distance obtained over that particular time interval.

On the contrary, we can instead vary two variables at once, such as both F_r and t_{thrust}. By evaluating the model with these parameters, keeping the angular force F_θ fixed and solving for the first 4000 seconds, we in fact obtain sets of values that give us a distance of less than $1m$.

A few of these are listed in Table 2 below, including the fuel consumption as defined previously in Section 2 (2.14).

F_r	F_θ	t_{thrust}	t_{min}	d_{min}	Fuel
-4.5	100	50	920.6	0.5269	5225
-1	100	25	1321.0	0.2879	2525
7	100	20	1483.9	0.1427	2140
31	100	250	383.9	0.4024	32750
32	100	266	377.0	0.8493	35112
34	100	310	365.1	0.2006	41540
34.5	100	330	362.5	0.0729	44385

Table 2: Table of Values where the debris is reached by the spacecraft. t_{min} is the time at which the minimum distance d_{min} for these values occurs.

From the data in the table the most advantageous case is the third as it uses the least fuel, despite taking the longest time to achieve this minimum distance. However, from the table it should be evident that we can segregate the first three cases from the last four as nearly every variable is significantly different.

To begin with we can categorise the first three cases by our determined value of F_r. These cases coincidentally correspond to low values of t_{thrust}, that is if we want the distance to be minimised. This is excellent since the consumption of fuel is greatly reduced as opposed to the four other cases. However, we notice that the minimum for each of these cases occurs over 15 minutes after the thrusting begins. The graphs for each of the first three cases are shown in Figure (1) below.

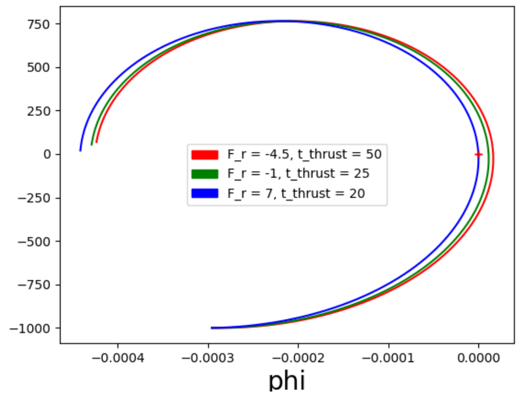


Figure 1: Graph of $z(t)$ against $\phi(t)$ for the first three cases in Table (2).

Moreover, we can also graph the spacecraft trajectories for the last four cases, as shown below in Figure (2). In this figure we notice that the radius of each orbit decreases as the radial force decreases. This makes sense logically as we would expect the orbit to be smaller with a smaller force applied. Yet with respect to Table (2), in comparison to the first few cases, both F_r and t_{thrust} are greater, implying a greater fuel consumption. In addition, each of the last four cases occur between 6 and 7 minutes after boosting, and in fact the last case reaches the closest to the debris.

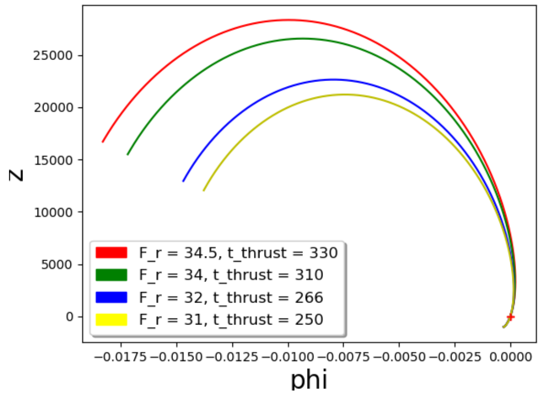


Figure 2: Graph of $z(t)$ against $\phi(t)$ for each of the last four cases in Table (2).

4.3 Retrospective Approach

Q6

Having just seen how the two orbital objects behave in different circumstances, we can look back on our findings and deduce an improved strategy for reaching the debris. From Table (2) the best case is when $F_r = 7$ and $F_\theta = 100$ as this uses the least fuel. However if the thrusters are not necessarily used from the start, then the best way to reach the debris is to wait until the spacecraft is a specific distance behind the debris so that as we increase F_r the craft approaches the debris from below. In this instance, we can fix $F_\theta = 0$, which reduces the amount of fuel consumed, that is $Fuel = |F_r| \cdot t_{thrust}$.

On the other hand, a more efficient approach would be to collect several pieces of debris at once. Not only does this path reduce the amount of fuel consumed, but it also reduces the time to clear the same quantity of debris.

If we assume that the speed of the gas particles released from the thrusters is $1km/s$ and all of this momentum is transferred to the ship, the craft will use $1kg$ of fuel per $1km$ travelled. Under a further assumption, that the spacecraft can carry $500kg$ of fuel at maximum capacity, the craft can collect debris between an altitude of $300km$ and $500km$. However, enough fuel must remain after collecting debris so that the craft can return to the lower atmosphere, otherwise the craft won't be able to return and in turn become debris itself. In reality these are all crude approximations, the gravitational forces have not been fully considered, this means that more mass of fuel is required to move away from the Earth, and vice versa.

5 Impact of Debris

Having discussed the possible threats of debris striking the surface in Section (1.1), we can begin to expand on the methodology built up in Section (2.5). However with the atmosphere greatly influencing the debris' trajectory, we must also consider kinetic molecular theory to model forces between atoms of gas.

5.1 Kinetic Molecular Theory

We can start by deriving the average speed of molecules in the atmosphere from the ideal gas law: $PV = nRT$. P is the pressure of the system, V is the volume, T is the system's temperature, n

represents the number of moles of gas, finally R is known as the ideal gas constant [10].

$$PV = \frac{2}{3}NK_{avg},$$

$$T = \frac{2N}{3nR}K_{avg},$$

$$K_{avg} = \frac{3k_B}{2}T.$$

N is the number of molecules, K_{avg} is the average kinetic energy of the molecules and $k_B = 1.381 \times 10^{-23} J/K$ is defined as the Boltzmann constant. Yet as we are working in three dimensions, we can define the average kinetic energy in one single direction simply as,

$$K_{avg} = \frac{k_B}{2}T. \tag{5.1}$$

In order to find the average speed of the molecules we can use the following equivalence, which is that,

$$K_{avg} = \frac{k_B}{2}T = \frac{1}{2}mv^2,$$

$$v = \sqrt{\frac{k_B}{m}T}. \tag{5.2}$$

At an altitude of $300km$ in the thermosphere, the temperature varies largely, even over a period of 24 hours [11], although for simplicity we can assume the temperature at this height is $1500K$. Instead at an altitude of $100km$ the temperature drops roughly to $300K$. We can also assume the density of the gas, $\rho = 10^{-11}kg/m^3$. Using nitrogen atoms and Avogadro's constant, $6.022\cdots \times 10^{23}mol^{-1}$, we can find the speed of these particles.

$$m_N = \frac{0.0140067}{6.02214086 \times 10^{23}} = 2.326 \times 10^{-26}kg,$$

$$v_N = \sqrt{\frac{1.381 \times 10^{-23}}{2.326 \times 10^{-26}} \cdot 1500} = 944\ m/s.$$

Comparing this to the velocity of an orbiting object at this height using equation (2.13),

$$v = \sqrt{\frac{6.673 \times 10^{-11} \cdot 5.97 \times 10^{24}}{6770 \times 10^3}} = 7670\ m/s.$$

Given that the difference between both speed values is very large we can assume that the molecules are stationary in comparison to the debris. In actual fact under this assumption, the collision between a stationary particle and the debris generates a large moment force as the mass of the particle is relatively negligible against the debris. Therefore, this is the same as the force lost by the piece of debris; the frictional force is then given by,

$$F_{fr} = 2A\rho_A v_h^2. \tag{5.3}$$

As expected, this frictional force is proportional to the surface area A of the debris, the density of the atmosphere ρ_A and the squared velocity v_h^2 of the debris at an altitude h.

5.2 Relating Energies

To extend the methods begun in Section (2.5) we can start again by summing the potential energy and kinetic energy of the debris, defining $r_h = R_E + h$, we have that,

$$E_h = P_h + K_h = -\frac{GmM_E}{r_h} + \frac{GmM_E}{2r_h}, \tag{5.4}$$
$$= -\frac{GmM_E}{2r_h}. \tag{5.5}$$

Using equation (2.13) we find the velocity of the debris in terms of its total energy E_h, Q7

$$v = \sqrt{\frac{-2E_h}{m}}. \tag{5.6}$$

From equation (5.5), we can now define the power of the debris via differentiation,

$$\frac{dE_h}{dt} = \frac{GM_Em\frac{dr_h}{dt}}{2r_h^2}. \tag{5.7}$$

Utilising equation (5.3), defined in the previous section as the frictional force, it follows that,

$$\frac{GM_Em\frac{dr_h}{dt}}{2r_h^2} = -2A\rho_A v_h^3. \tag{5.8}$$

Rearranging this equation and taking the direction towards the Earth to be positive, we find that,

$$\frac{dr_h}{dt} = \frac{4A\rho_A v_h^3 r_h^2}{GM_Em}. \tag{5.9}$$

Now, by once again using (2.13), (5.9) reduces to,

$$\frac{dr_h}{dt} = \frac{4A\rho_A\sqrt{GM_Er_h}}{m}. \tag{5.10}$$

5.3 Introduction of Atmospheric Density

We have already used the idea of this density in the frictional force, however this value now becomes very important. The density of the atmosphere varies according to altitude, but influences the energy possessed by the debris. The value of the atmospheric density can in fact be explicitly approximated as a function of h,

$$\rho_A(h) = ae^{-hl} + Bh^{-\sigma}. \tag{5.11}$$

As the density of the atmosphere varies wildly, this equation is only a rough guide; we can determine the constants a, l, B and σ from known data. That is,

$$a = 1.946, \quad l = 1.5 \times 10^{-4}, \quad B = 4.63 \times 10^{30}, \quad \sigma = 7.57. \tag{5.12}$$

These values can now help us in solving (5.10), in order to do this we can assume that the altitude is very small, *i.e.* $r_h \approx R_E$ and $\rho_A \approx Bh^{-\sigma}$. In reality this is not true, but only gives a small margin of error. Hence we can solve this first-order separable differential equation,

$$\frac{dr_h}{dt} = \frac{dh}{dt} = \frac{4ABh^{-\sigma}\sqrt{GM_E R_E}}{m}, \tag{5.13}$$

$$\int_0^{h_0} h^\sigma \; dh = \int \frac{4AB\sqrt{GM_E R_E}}{m} \; dt, \tag{5.14}$$

$$\frac{h_0^{\sigma+1}}{\sigma+1} = \frac{4AB\sqrt{GM_E R_E}}{m} \cdot t \quad (+\, constant\,). \tag{5.15}$$

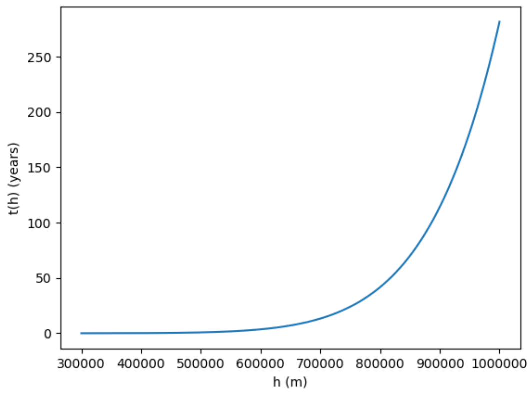
If we fix $h_0 = 0$, where h_0 is the altitude to begin with, it makes sense to say that the time to enter the atmosphere is zero, *e.g.* $t = 0$ so upon substituting this in, the constant vanishes, eventually concluding that,

$$\boxed{t = \frac{h_0^{\sigma+1} m}{4(\sigma+1)AB\sqrt{GM_E R_E}}.} \tag{5.16}$$

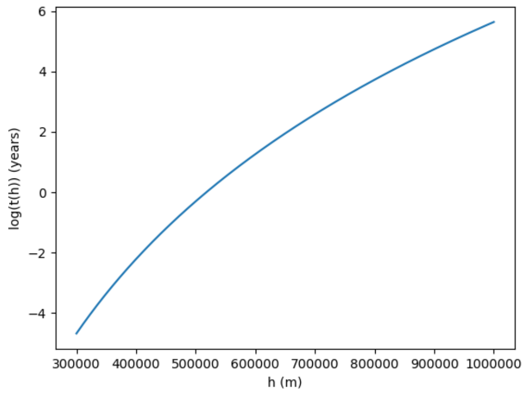
5.4 Examples of Debris

In a few examples of different types of debris below, we will vary the starting altitude h_0 between $300km$ and $1000km$ to see how long it takes in each case for the debris to fall back to Earth.

Example 1: A cube of aluminium has density $\rho = 2700kg/m^3$ and volume $V = 1\ \mu m^3$, with mass $2.7g$ the ratio $\frac{m}{A} = 27$ using SI units. Now using (5.16) and the values of the constants in (5.12), we can graph the time ($years$) as a function of initial height (m).



(a)

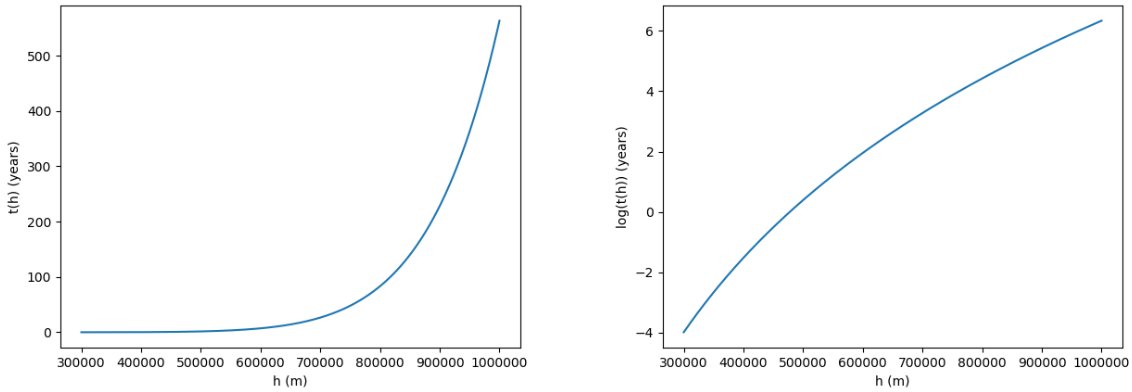


(b)

Figure 3: (a) A graph showing the time in which, for an initial height h, a cube of aluminium takes to fall to Earth. **(b)** Semi-Logarithmic graph of (a).

Example 2: This time a rod of aluminium has length $L = 10cm$, and width $l = 1cm$. The average of each surface, $A = L \times \frac{l}{2}$ and has mass $m = l^2 L\rho$. Therefore the ratio $\frac{m}{A} = 2\rho l =$

$2 \times 2700 \times 0.01 = 54$. By using the same method as above, we obtain both graphs in Figure (4) below.

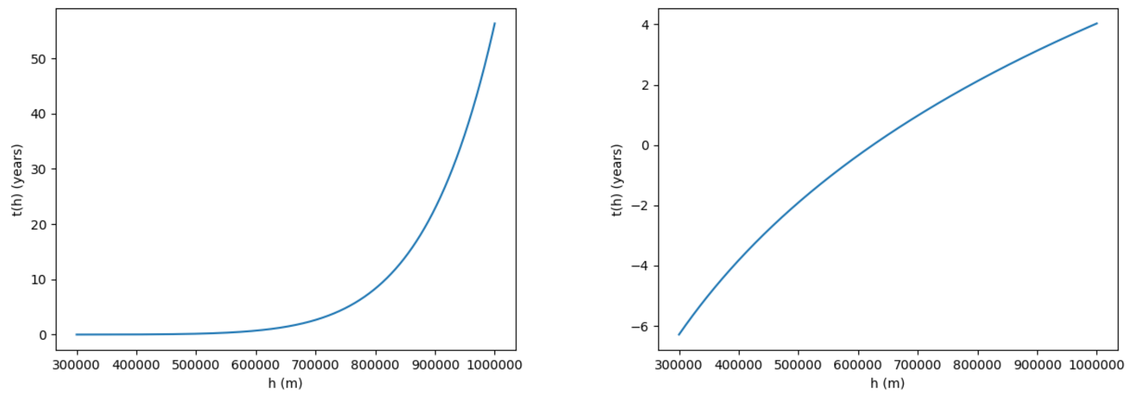


(a) (b)

Figure 4: **(a)** A graph showing the time in which, for an initial height h, a rod of aluminium takes to fall to Earth. **(b)** Semi-Logarithmic graph of (a).

This is in fact very similar to the first example, however one can notice that the mass-area ratio is in fact double. By observing the graphs, the scales for **(a)** have also doubled. That is for any given initial height, the time to fall back to Earth is roughly twice that of Example 1.

$\boxed{\textbf{Example 3:}}$ A square aluminium plate has length $L = 10cm$ and width $l = 1mm$. The average area of the plate is $A = L^2/2$. It's mass $m = L^2 l\rho = 27g$, with ratio $\frac{m}{A} = 5.4$.

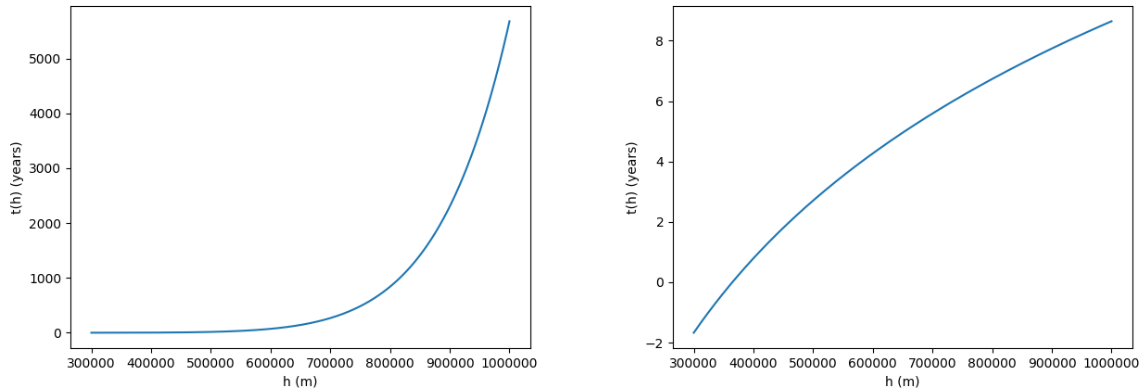


(a) (b)

Figure 5: **(a)** A graph showing the time in which, for an initial height h, a flat, square plate of aluminium takes to fall to Earth. **(b)** Semi-Logarithmic graph of (a).

The first three examples we have just considered are all relatively light, that is the ratios $\frac{m}{A}$ are small, however debris can often consist of much more dense objects, such as broken spacecrafts.

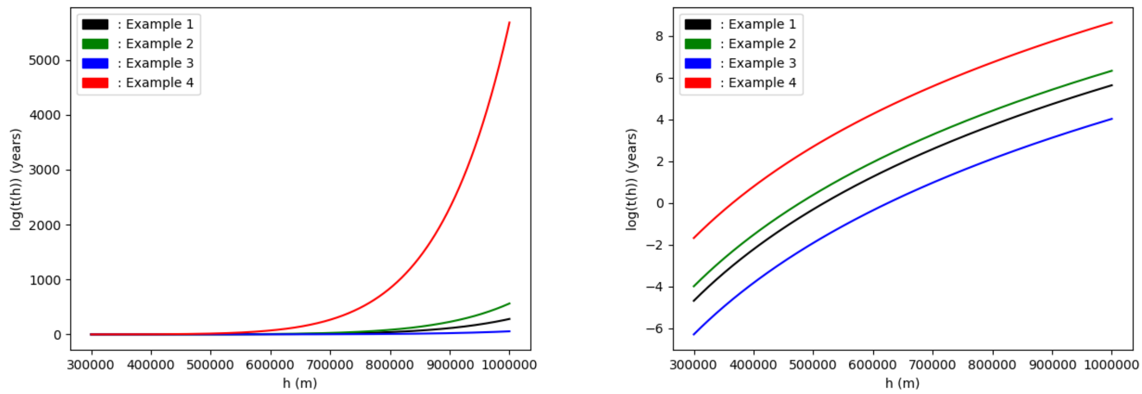
$\boxed{\textbf{Example 4:}}$ The Gemini spacecraft was NASA's second-ever human spaceflight program,, which began in 1961. The craft had an average mass of $3850kg$ and for simplicity can assume that it's shape was a sphere of diameter $3m$. This implies that the area is always $\pi \cdot (\frac{3}{2})^2 = 7.068\ldots$, regardless of the direction it travels, therefore the ratio $\frac{m}{A} = 544.663\ldots$.



(a) (b)

Figure 6: **(a)** A graph showing the time for which, for an initial height h, the Gemini spacecraft falls to Earth. **(b)** Semi-Logarithmic graph of (a).

In general these graphs are very hard to compare, since the only visible difference is the scaling on the y-axis. We can therefore give a comparison between the two types of graphs on the same axes.



(a) (b)

Figure 7: Comparison graphs for each of the four examples above.

The most noticeable difference is on Figure (7), where the slope in graph **(a)** for the fourth example increases much more dramatically, after $700km$, than any of the others. In relation to the ratio $\frac{m}{A}$ for each case, this correlates. The value $544.663\ldots$ is much larger than for any of the other cases used, meaning that the time for the debris to impact Earth is much greater at larger starting altitudes. In fact this also applies to the other cases, that is the value $\frac{m}{A}$ correlates positively with the position and slope of the plots.

The data provided by NASA [12] also agrees with our findings. For example at an altitude of $500km$ we can measure the time in decades, whilst in Example 4 we find that at this height the time taken is roughly 15 years. Similarly, most examples at an altitude of $1000km$ do in fact remain in orbit for centuries or longer.

6 Conclusion

Once we begin to numerically describe our system it becomes much more manageable. We can then logically deduce particular quantities about the system that are crucial to our model. By introducing Newton's laws we were able to algebraically describe the way in which two orbital objects interact about a celestial body. We then solved these equations numerically using the Runge-Kutta 4th order method, further modelling a spacecraft, implementing the idea of fuel and thrusters into our system. As in many cases, the debris fell back down to Earth and as a consequence we can model how the debris interacts with atmospheric molecules.

With the main goal for this report concerned about the ramifications as a result of debris, we have explored a few approaches that prevent damage and reduce the quantity of debris. One of the primary techniques we explored was an active spacecraft.

By simulating the orbits of this spacecraft and a piece of debris, we found that we could optimise the distance between the two objects so that the craft would use as little fuel as possible. In fact if we fixed the force due to the thrusters to $7N$ radially outwards and $100N$ parallel to the Earth, the craft would reach within $1m$ of the debris with minimal fuel consumption. Moreover, after the inclusion of atmospheric density, we observed a few examples of debris that would eventually fall to Earth. We noticed that the time it took was firstly dependant on it's initial altitude, but also that as the ratio $\frac{\text{mass}}{\text{area}}$ increased, the time taken to fall correlated positively.

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