Modelling the Behaviour of Lightning

MATHEMATICAL MODELLING ASSIGNMENT

Alex Rowan-Smith | **Practical Group: 4**
November 2018 | **CISid: nvkl56**

ABSTRACT

Lightning and thunder clouds form an integral part of the climate in many areas across the world, becoming more prevalent due to an increasing rate in global warming. They produce numerous detrimental effects to civilisation, particularly in poorer regions without the capabilities to provide necessary protection to inhabitants. In this essay we aim to model the behaviour of lightning using computer simulations, numerical analysis and mathematical formulae; with the ultimate goal to prevent and mitigate effects of such lightning strikes. We will use various packages included in Python 3.7 to model a system, which we can use to investigate how a lightning channel permeates through a medium of air, as well as the interaction of lightning with positively charged objects.

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0.1 Introduction

Lightning is a natural phenomenon that can occur almost anywhere in the world. It is an electrostatic discharge which is caused by a potential difference between a cloud and another surface. In this report we will aim to model and study the effects of a particular type of lightning, known as *CG lightning* or *forked lightning*, this is where a potential difference is induced between the clouds and the earth. As a side note, there are other types of lightning, for example between two clouds, however our focus will remain on forked lightning. [1]

One of the most crucial conditions for lightning to form is the humidity of the air and temperature of the ground. The temperature needs to be relatively hot as this causes warm air to rise and contributes to the development of the cloud. At the top of the cloud, positive charges begin to build up due to a huge temperature difference between the top of the cloud and the bottom. At the same time ice crystals begin to form, an electrostatic repulsion strengthens as a result of collisions and friction between these ice crystals. Now there is an increasing potential difference within the cloud as negative charges fall rapidly, once this difference in potential becomes sufficiently large, static electricity starts to form. Since the charge at the bottom of the cloud is overwhelmingly negative, a potential difference is directly set up between the cloud and the ground, which hence, must be positively charged. Lightning formed in this way is negatively charged as there is a net transfer of negative charge towards the ground, and therefore attracted to the closest *positive* charge on the surface.

As warmth is need for the formation of thunder clouds, lightning becomes particularly concentrated in regions between the tropics. This is a massive problem for civilisation since lightning is attracted to the closest positively charged object on the earth, whether a building, tree, or even a person. The severity of this problem is only intensified by the fact that the temperature of lightning is close to 30000K! However we can often use methods, like lightning rods that protect us from this phenomenon [2]. We shall endeavour to model this later on in Section (3.1).

1.1 Formulating the Model

To begin to create a model we need to describe the system we are going to work with. We want to set up a lattice of points (nodes) where the first row represents the cloud and the last representing the earth. In an ideal situation we initiate a lattice with infinite width, however this is not easily implemented. We therefore take the width of the lattice sufficiently large, such that the likelihood of the channel extending out of our range is minimal. As in Figure (2) we optimise the width of our lattice as 100 nodes; taking any larger width will dramatically increase the compilation time.

The underpinning methodology for our model consists of adding a new node to the channel and then updating the potential according to this new channel as seen below in Section (1.1.3), then further iterating over a specified number of steps.

1.1.1 Describing the Quantities in our Model

One important aspect of the model is the electric potential, denoted by ϕ, which we, in terms of our lattice, can define as follows:

$$\phi_{i,j} = \phi(x = jh, y = ih)$$

This is the electric potential at the point (i, j) where we can construct h as the spacing between two adjacent nodes. To clarify, $\phi_{0,j} = 0$ and $\phi_{N-1,j} = 1$ represent all nodes in the cloud, and at the ground respectively, that is if N is the height dimension of our lattice. In our model it is only possible to work with approximations as we are restricted to using only a finite quantity of lattice nodes.

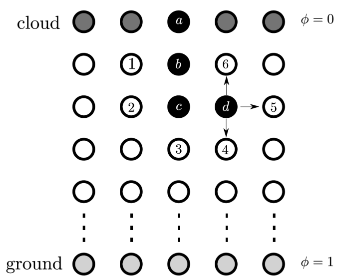


Figure 1: Visual representation of the lattice. Black nodes show the path of the channel, whereas the numbered white nodes show the possible path the channel can take at the next stage. This stochastic process involving electric potentials is known as a Wiesmann-Zeller model. [3]

1.1.2 Assignment Problem 2

Q2

From our model we can assume that $\phi(x, y = y_{cloud}) = 0$ and $\phi(x, y = y_{ground}) = 1$. The potential can in fact be initiated elsewhere by the Poisson equation,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \phi(x, y) = 0. \tag{1.1}$$

For detail we treat the electric potential between the clouds as a uniform field, and fix the width of the model large enough so we do not concern ourselves with an edge effect. This implies the potential is only dependant on the displacement from the cloud, so we can say,

$$\phi_{i,j} = \frac{i}{N}, \quad \forall \quad i, j \in [0, N]. \tag{1.2}$$

Equation (1.2) gives us a trivial solution to the Poisson equation (1.1), assuming that no lightning has occurred yet, as this would disrupt the uniform field in our assumption.

1.1.3 Implementing the Jacobi Relaxation Method

However, once we introduce the lightning into our model we must describe the way in which the field changes due to the introduction of negative charges, *i.e.* nodes that the lightning channel has already reached. We can solve this numerically using a technique called the Jacobi relaxation method. This method is extremely useful in our model as we need to solve hundreds of equations simultaneously. We can encode the following formula into our model to update each value of potential at each node.

$$\phi_{i,j}^{n+1} = \frac{1}{4} \left(\phi_{i+1,j}^{n} + \phi_{i-1,j}^{n} + \phi_{i,j+1}^{n} + \phi_{i,j-1}^{n} \right) \tag{1.3}$$

$\phi_{i,j}^{n+1}$ is now the updated version of the node $\phi_{i,j}^{n}$ according to the Jacobi method. We shall then iterate this for all nodes in the lattice. Once (i, j) is set as an element of the channel we must redefine $\phi_{i,j} = 0$.

2.1 Initiating an Algorithm for Lightning

Having now updated all nodes in the space, we need to begin to consider how lightning behaves. This involves how the channel travels in the lattice, as well as why it takes a specific path.

2.1.1 Random Nature

It is in fact not necessarily naïve to suggest that we can use randomness in our model. As in Figure (1) if we take the node d as an example, the possible nodes that the lightning channel can extend to are $4, 5$ or 6. Since the potential difference between d and the adjacent nodes varies, we can relate this voltage to the probability of the adjacent node being chosen, assuming that only one of $\{4, 5, 6\}$ can be chosen in any one iteration.

A pair of nodes with a greater potential difference are much more likely be chosen, by the minimum total potential energy principle [4] the channel will travel down the route of least work. Thus the probability that this node is chosen will be greater; this is also true vice-versa, *i.e.* a smaller potential difference is less likely. So we can now conclude,

$$\mathbb{P}(\phi_{i',j'}) \propto \mid \phi_{i',j'} - \phi_{i,j} \mid . \tag{2.1}$$

Actually we can go one step further and suggest a proportionality constant N that normalises the probabilities. Let $N \in \mathbb{R}$ *s.t.*

$$\sum_{(i',j')} \mid \phi_{i',j'} - \phi_{i,j} \mid \; = N \tag{2.2}$$

$$\implies \mathbb{P}(\phi_{i',j'}) = \frac{\mid \phi_{i',j'} - \phi_{i,j} \mid}{N}. \tag{2.3}$$

NOTE : In (2.2) we sum over all coordinates (i', j') which are adjacent nodes to (i, j) and are not already part of the channel.

2.1.2 Assignment Problem 9

Q9

Considering all nodes in the lightning channel have $\phi = 0$, the probability that a new node will be added laterally increases as the channel gets closer to the ground. As previously stated this is due to a larger potential difference between nodes, which, logically makes sense since the ground is also at zero potential (minimum potential in our model). Therefore nodes are more likely to branch rather than continue down the leading channel by (2.1).

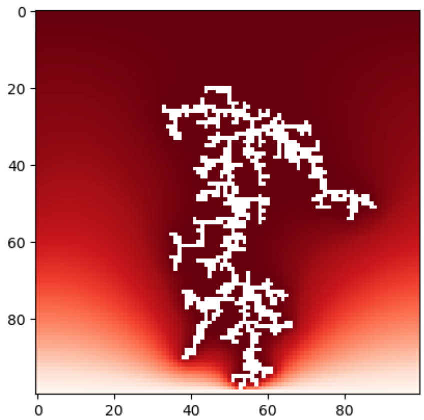


Figure 2: Plot of lightning channel produced from our model. $\phi = 1$ (Red), $\phi = 0$ (White)

Our conclusions are supported by analysing Figure 2, as the channel reaches the ground we find more branches. Moreover by the contrast of colours on the graph we can see there is a greater difference in potential laterally rather than vertically.

3.1 Protection from Lightning

One of the most lightning-struck countries on the planet is Venezuela. As aforementioned, humid climates are much more susceptible to lightning which is no surprise here as Venezuela lies just above the equator. In fact one area of the country has the highest number of recorded flashes per square kilometre per year at 250 [5]. However it is evident that this phenomenon poses a substantial threat to humanity. Consequently it is paramount to their civilisation that people, nature and structures are protected as well as possible. Before we explore approaches to mitigation, we shall first examine certain consequences of a lightning strike.

3.1.1 Effects of Lightning Strikes

Beginning with perhaps the most obvious and direct effects of lightning, namely electrodynamic effects. A strike can tear apart buildings and almost anything with a high conductivity or low impedance. This is also true for trees, due to the intense temperature of lightning forest fires can catch on rapidly particularly in dense regions.

Figure 3: Forest fire due to lightning strikes in Alberta, Canada. [6]

On the other hand we can also observe indirect effects as a result of lightning. One notable example is the impact on a power grid, which can cause surges in the power lines. [7]

As a matter of fact, most objects protruding from the earth's surface are in danger from lightning. This includes mountain ranges, which despite general inaccessibility pose a great threat to mountaineers and hikers. [8]

3.1.2 Mitigation and Prevention Techniques

Despite the destructive nature of lightning, we can employ techniques that attempt to reduce the severity of a strike. As a more general rule, avoiding conductive metals and moving away from protruding objects is always safe. Yet one very successful method is a lightning conductor, these rod-like objects aim to conduct the electricity and do so by using a highly conductive material. We compare numerous common metals by conductivity below.

Material	**Conductivity** ($\times 10^6$ $^S/_m$)
Silver	62.1
Copper	58.7
Aluminium	36.9
Brass	15.9
Steel	10.1

Figure 4: Table showing the conductivity of various metals and alloys. [9]

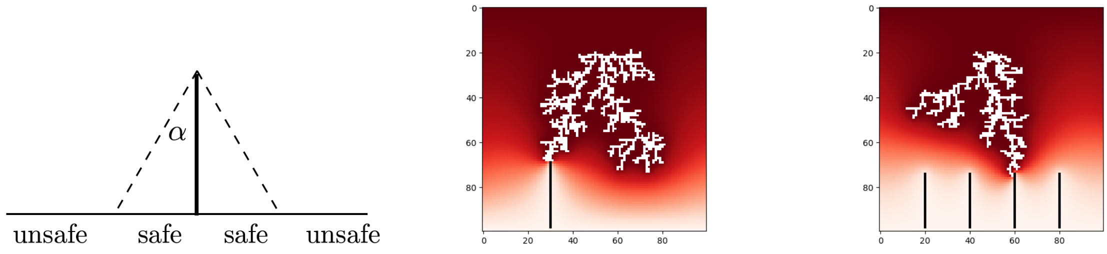
Often lightning conductors are made of copper or aluminium, which is supported by Figure 4 as both materials have a relatively high conductivity and are both cheap in comparison to materials with a slightly higher conductivity.

On the other hand, a single lightning conductor has a small range of protection so are not always beneficial for structures or buildings with a large surface area. Instead many large buildings have lightning protection systems installed, which are in some places a legal requirement. Such a system is comprised of a network of rods around a building, connected via large conductor cables which transfer the energy of the strike into the ground safely.

The 'rolling-sphere' method is a simulation that can be used in order to identify any susceptible points on a structure, generally the sphere has a radius of 150ft to ensure the building is well protected. At any points marked on the building where the sphere makes contact, air terminals are placed, hence the sphere can then not touch the building. [10]

3.1.3 Implementing a Lightning Rod in our Model

In order to model a lightning rod in our system, we can safely assume that the potential at all nodes contained in the rod are 1 since, logically, lightning travels to the nearest conductor due to the greatest potential difference as explained previously in Section 2.1.



(a) (b) (c)

Figure 5: **(a)** Angle α of protection from lightning rod. **(b)** Output of our model, including lightning rod. Red represents nodes at a potential of 1 and white of 0. **(c)** Output involving a network of rods.

This is shown in the plot of our model, Figure 5(b). The channel branches randomly, yet we can see that the branches closest to the rod converge at the tip.

Using the same method, we can extend the number of rods to represent how a lightning protection systems work on a large building. As above in Figure 5(c) we can model a network of rods instead of a singular rod. Shown by the whitish region in (c) it is clear that any structure below is well-protected as long as the electricity produced is earthed. This is in comparison to (b), where the channel still has a chance of striking the building below.

4.1 Modelling Using Point Charges

Modelling the rod as in Section 3.1.3 can actually be simplified. If we take the top node of the rod to be a point charge, the nodes below can be suitably ignored. By using a point charge we do not need to concern ourselves with the dimension of the rod. This further benefits us since now the equation for potential can be written in a third dimension, as explained below.

4.1.1 Assignment Problem 13

In order to investigate the potential further, we can use the Laplacian of the potential, which we can now define against the Dirac delta function, which is itself defined by $\int \delta(r) f(r) dr = f(0)$, Q13

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$$\nabla^2\phi(x_1,x_2,x_3) = \frac{\partial^2\phi}{\partial x_1^2} + \frac{\partial^2\phi}{\partial x_2^2} + \frac{\partial^2\phi}{\partial x_3^2} = \delta(r)q \quad where \quad r = \sqrt{x_1^2+x_2^2+x_3^2}. \tag{4.1}$$

In order to simplify our calculations, we can use the Einstein Summation Convention, which in our situation means that, $\sum_{j=0}^{3} x_j = x_i$.

$$\frac{\partial r}{\partial x_i} = \frac{x_i}{r}\,; \qquad \frac{\partial^2 r}{\partial x_i^2} = \left(\frac{1}{r} - \frac{x_i^2}{r^3}\right). \tag{4.2}$$

Then by the chain rule for partial derivatives:

$$\frac{\partial\phi}{\partial x_i} = \frac{\partial\phi}{\partial r}\frac{\partial r}{\partial x_i} \implies \frac{\partial^2\phi}{\partial x_i^2} = \frac{\partial^2 r}{\partial x_i^2}\frac{\partial\phi}{\partial r} + \frac{\partial r}{\partial x_i}\frac{\partial}{\partial x_i}\left(\frac{\partial\phi}{\partial r}\right). \tag{4.3}$$

Using the fact that $\frac{\partial}{\partial x_i} = \frac{\partial}{\partial r}\frac{\partial r}{\partial x_i}$, (4.3) is equivalent to:

$$\frac{\partial^2\phi}{\partial x_i^2} = \frac{\partial^2 r}{\partial x_i^2}\frac{\partial\phi}{\partial r} + \left(\frac{\partial r}{\partial x_i}\right)^2\frac{\partial^2\phi}{\partial r^2}. \tag{4.4}$$

By substituting in the formulae for the partial derivatives of r with respect to x_i as in (4.2),

$$\frac{\partial^2\phi}{\partial x_i^2} = \left(\frac{1}{r} - \frac{x_i^2}{r^3}\right)\frac{\partial\phi}{\partial r} + \left(\frac{x_i}{r}\right)^2\frac{\partial^2\phi}{\partial r^2}. \tag{4.5}$$

Now from the definition and the Einstein notation we want to sum this equation over all values of i, we need to also use the fact that $r^2 = x_1^2 + x_2^2 + x_3^2$, defined in (4.1),

$$\begin{aligned}\nabla^2\phi(x_1,x_2,x_3) &= \left(\frac{3}{r} - \frac{r^2}{r^3}\right)\frac{\partial\phi}{\partial r} + \left(\frac{r^2}{r^2}\right)\frac{\partial^2\phi}{\partial r^2}\\ &= \frac{1}{r^2}\left[2r\frac{\partial\phi}{\partial r} + r^2\frac{\partial^2\phi}{\partial r^2}\right]\\ &= \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial\phi}{\partial r}\right] = \delta(r)q. \end{aligned}\tag{4.6}$$

While at first this equation may seem more complicated, it becomes very important for our calculations in Assignment Problem (14) where we can find an explicit equation for the potential ϕ.

4.1.2 Assignment Problem 12

Q12

As demonstrated in Figure 5(a) the lightning conductor protects anything in a certain radius of the rod. Now transitioning to Figure 5(b) it is fairly clear that there is in fact a cone around the rod where the potential is reduced, relating to the 'protection zone': what the area is more commonly referred to as. [11]

From Gay-Lussac [12], if we assume that the radius of the protection zone is double the height of a lightning rod, we can find a suitable α, that is $\tan^{-1}(2) \approx \alpha = 63.4^o$ *(3 s.f.)*. However often the angle is based upon perception, for example if we do several calculations on Figure 5 (b), we can give a rough estimation of α in a range between 45.0^o and 26.6^o. Although these estimates don't coincide with Gay-Lussac, there are other factors to consider that affect this range, such as the height and placement of the rod. Indeed from results and further citations in [11] and [13], we can be reassured by the value of our results, since many of the values attained by real experiments lie in a region of 70.0^o and 20.0^o.

4.1.3 Assignment Problem 14

Q14

From the Dirac delta function $\delta(r)q$ in equation (4.1), it can be shown that,

$$\int_V \delta(r)q \, dV = q. \tag{4.7}$$

Furthermore the integral of the left-hand side of equation (4.1) can also be shown to reduce to,

$$\int_V \nabla^2\phi(r) = \lim_{R\to\infty} 4\pi R^2 \frac{\partial\phi(r)}{\partial r}(R), \tag{4.8}$$

over the whole domain by an integration of laplacian theorem.
Rearrangement of the equation (4.6) gives us,

$$\phi'' + \frac{2}{r}\phi' = \delta(r)q, \tag{4.9}$$

in which we can attempt to solve using differential equations.
Substituting,

$$u' + \frac{2}{r}u = \delta(r)q \implies f(r) = \frac{2}{r},\ g(r) = \delta(r)q. \tag{4.10}$$

Now since this is a first-order linear differential equation, we can find an integrating factor,

$$I(r) = e^{\int f(r)\, dr} = r^2 \tag{4.11}$$

$$\begin{aligned} \implies u(r) &= \frac{1}{r^2}\int r^2\delta(r)q\, dr \\ &= \frac{1}{r^2}\left(r^2 q - 2\int qr\, dr\right) \\ &= \frac{1}{r^2}\left(r^2 q - qr^2\right) = 0 \end{aligned} \tag{4.12}$$

From this we can conclude (4.1) $= 0$, that is

$$\begin{aligned} &\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) = 0, \implies r^2\frac{\partial\phi}{\partial r} = c_1, \quad \text{for some } c_1 \\ &\implies \frac{\partial\phi}{\partial r} = \frac{c_1}{r^2}, \implies \phi = c_2 - \frac{c_1}{r}. \end{aligned} \tag{4.13}$$

In order to determine the constants c_1 and c_2 we can say that $\lim_{r\to\infty}\phi(r) = 0$,

$$\lim_{r\to\infty} c_2 - \frac{c_1}{r} = 0 = c_2 \tag{4.14}$$

Now by equating (4.7) and (4.8) we can determine the final constant c_1,

$$\begin{aligned} q &= \lim_{R\to\infty} 4\pi R^2\frac{\partial\phi(r)}{\partial r}(R) \\ &= \lim_{R\to\infty} 4\pi R^2\frac{c_1}{R^2} = 4\pi c_1 \end{aligned} \tag{4.15}$$

This implies that $c_1 = \frac{q}{4\pi}$, substituting this back into equation (4.13) gives us a final equation for the potential:

$$\phi(r) = \frac{\left(-\frac{q}{4\pi}\right)}{r} = -\frac{q}{4\pi r}. \tag{4.16}$$

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4.1.4 Assignment Problem 15

Q15

Up to now we have modelled the potential of our system as a linear equation ranging from 0 to 1, yet this is generally unrealistic. Moreover we can use a new formula $\phi_{cloud}(y = H) = \sigma_g H_y$, where σ_g is the electric charge density at the ground. Assuming that our model acts similar to a capacitor, we know that the electric field strength E is uniform from equation (1.2) and can be written like so:

$$E_y = \frac{V_y}{H}; \tag{4.17}$$

such that H is the vertical distance from the cloud to the ground, and V_y is the potential difference between a point y and the cloud.

In this model the cloud still has a potential of 0, $\phi_{cloud}(y = 0) = 0$. This implies that we can describe the potential difference as follows.

$$V_y = |\phi_{cloud}(y = H) - \phi_{cloud}(y = 0)| = |\phi_{cloud}(y = H)| \tag{4.18}$$

$$\implies V_y = \sigma_g H_y, \implies E_y^{cloud} = \frac{\sigma_g H_y}{H_y} = \sigma_g \tag{4.19}$$

By modelling the top of the rod as a sphere (with radius R), we can use Coulomb's law to describe the potential at a distance r from the centre. In fact it follows that the charge is uniform across the surface of the sphere, *i.e.*

$$q = 4\pi R^2 \sigma_r, \tag{4.20}$$

where σ_r is the charge density on the surface of the sphere. From equation (4.16) this implies that

$$\phi^{rod}(r) = -\frac{4\pi R^2 \sigma_r}{4\pi r} = -\frac{R^2}{r}\sigma_r. \tag{4.21}$$

Having calculated the potential as a function of r, we can find the point between the clouds and the rod where the electric field strengths are equal. This is also the point where a given charged particle is in equilibrium, as no force acts on it. The minus sign in the equation above only represents the direction since we are dealing with vectors, however for comparison can be suitably ignored.

$$E_r^{rod} = \frac{\partial \phi(r)}{\partial r} = \frac{q}{4\pi r^2} = \frac{R^2}{r^2}\sigma_r \tag{4.22}$$

Combining this with (4.19) we get that,

$$\sigma_g = \frac{R^2}{r^2}\sigma_r \implies r = R\sqrt{\frac{\sigma_r}{\sigma_g}}. \tag{4.23}$$

Furthermore, we are able to find an equation relating σ_r and σ_g by equating the two potentials. Indeed it can be shown that $\phi^{rod}(r = R) = \phi^{cloud}(y = H)$. From (4.21), we take $r = R$ as we are working on the surface,

$$\sigma_g H = R\sigma_r \implies \sigma_r = \sigma_g \frac{H}{R} \tag{4.24}$$

As an example, taking $H = 500$m and $R = 0.001$m, we can give a measure of the ratio between charge densities.

$$\frac{\sigma_g}{\sigma_r} = \frac{R}{H} = \frac{0.001}{500} = 2 \times 10^{-6} \tag{4.25}$$

However it is indeed possible to model the lighting channel as a point charge, where we can view the resultant electric field as the sum of two charges. From Coulomb's law, there exists an electrical force between two charges over a given distance. This suggests we can obtain a better insight into how the rod 'attracts' lightning.

5.1 Conclusion

Once we begin to numerically describe our system it becomes much more manageable. We can then logically deduce particular quantities about the system that are crucial to our model. By using techniques like the Jacobi-relaxation method and previously formed models such as the Wiesmann-Zeller model, we are able to consolidate variables from our system and form numerical equations. This then assists us in coding an accurate model, but with only finite precision due to approximations in these techniques.

With the main goal for this report concerned on protection and mitigation, we have explored a few approaches to prevent damage and nullify effects. One of the primary techniques we explored was the lightning rod, which we could impose into our model in order to properly study the behaviour of lightning in contact with a positively charged rod. As a matter of fact, in our findings, it was evident that the channel converges to the top node contained in the rod.

By modelling the rod as a point charge, we can more accurately calculate the potential in three dimensions. In fact to further this study, we could also model the lightning channel as a point charge and observe the forces between them as a result of Coulomb's Law.

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